CECH-COMPLETENESS IN FIBREWISE TOPOLOGY(General Topology, Geometric Topology and Their Applications)

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1. Introduction

We introduce a new notion "Čech-complete map", and investigate some basic properties, invariance under perfect maps, relationships between (locally) compact map, Čech-complete map and k-map, and characterizations by compactifications of Čech-complete maps.

Motivation and significance of Čech-complete maps are:

(1): The notion of Čech-complete maps in the fibrewise category TOP_B is corresponding to the notion of Čech-complete spaces in the topological category TOP. In fact, we can prove the following:

Compact map $\Rightarrow$ Locally compact map $\Rightarrow$ Čech-complete map $\Rightarrow$ k-map.

(2): The notion of Čech-complete maps is a new idea in General Topology. So, General Topology becomes plentifully by this notion.

For an arbitrary topological space B one considers the category TOP_B, the objects of which are continuous maps into the space B, and for the objects $p : X \to B$ and $q : Y \to B$, a morphism from p into q is a continuous map $\lambda : X \to Y$ with the property $p = q \circ \lambda$. This is denoted by $\lambda : p \to q$. A morphism $\lambda : p \to q$ is said to be onto, closed, perfect, etc., if respectively, such is the map $\lambda : X \to Y$. A continuous map $p : X \to B$ is called by a *projection*, and X is called by a *fibrewise space over* B or (X, p) is called by a *fibrewise space*. Further we call $\lambda : X \to Y$ a *morphism* when we write $\lambda : p \to q$, and we also call it a *fibrewise map* when we write $\lambda : (X, p) \to (Y, q)$.

Throughout this paper, we assume that all spaces are topological spaces, and all maps and projections are continuous. For other terminology and notations undefined in this paper, one can consult [3] about TOP, and [5] and [8] about TOP_B.

2. Preliminaries

In this section, we refer to the notions and notations in Fibrewise Topology.

Let (B, τ) be a fixed topological space B with a fixed topology τ. Throughout this paper, we will use the abbreviation *nbd(s)* for *neighborhood(s)*. We also use that for $b \in B$, $N(b)$ is the set of all open nbds of b, and $\mathbf{N}$, $\mathbf{Q}$ and $\mathbf{R}$ are the sets of all natural numbers, all rational numbers and all real numbers, respectively. Note that regularity of (B, τ) is assumed in Theorems 3.7 and 3.10, further first countability of (B, τ) is assumed in Theorem 3.10.

For a projection $p : X \to B$ and each point $b \in B$, the *fibre* over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset B' of B we regard $X_{B'} = p^{-1}(B')$ as a fibrewise space over B' with the projection determined by p. For a filter (base) $\mathcal{F}$ in X, we denote that $p_*(\mathcal{F})$ is the filter generated by the family $\{p(F) | F \in \mathcal{F}\}$.

First, we begin with defining some separation axioms on maps.

Definition 2.1. A projection $p : X \to B$ is called a *T_i-map*, $i = 0, 1, 2$, if for all $x, x' \in X$ such that $x \neq x'$, $p(x) = p(x')$ the following condition is respectively satisfied:

(1) $i = 0$: at least one of the points x, x' has a nbd in X not containing the other point;
(2) $i = 1$: each of the points x, x' has a nbd in X not containing the other point;
(3) $i = 2$: the points x and x' have disjoint nbds in X.

Definition 2.2. The subsets F and H of the space X are said to be respectively:

(1) *nbd separated* in $U \subset X$,
(2) *functionally separated* in $U \subset X$,

if the sets $F \cap U$ and $H \cap U$

(1) have disjoint nbds in U,
(2) are functionally separated in U (i.e. there exists a map $\phi : U \to [0, 1]$ such that $F \cap U \subset \phi^{-1}(0)$ and $H \cap U \subset \phi^{-1}(1)$).

Definition 2.3. A projection $p : X \to B$ is called *completely regular* (resp. *regular*), if for every point $x \in X$ and every closed set F in X such that $x \notin F$, there exists a nbd $W \in N(p(x))$, such that the sets $\{x\}$ and F are functionally separated (resp. nbd separated) in X_W. A completely regular (resp. regular) T_0-map is called *Tychonoff* or *$T_{3\frac{1}{2}}$- (resp. T_3-) map*.

It can be easily verified that every T_j-map is a T_i-map for $j, i = 0, 1, 2, 3, 3\frac{1}{2}$ and $i \leq j$.

Definition 2.4. Let $p : X \to B$ be a projection.
(1) The map p is called a *functionally T_2-map* if for all $x, x' \in X$ such that $x \neq x'$, $p(x) = p(x')$ there exists $W \in N(b)$ such that the sets $\{x\}$ and $\{x'\}$ are functionally separated in X_W.

(2) The map p is called *functionally normal* (resp. *normal*) if for every $b \in B$ and every two disjoint, closed sets F and H in X, there exists $W \in N(b)$ such that F and H are functionally separated (resp. nbd separated) in X_W. A functionally normal (resp. normal) T_3-map is called a *functionally T_4-map* (resp. *T_4-map*).

It can be easily verified that (1) every T_4-map is a T_3-map, (2) every functionally T_4-map is a $T_{3\frac{1}{2}}$-map and every $T_{3\frac{1}{2}}$-map is a functionally T_2-map. However, note that every T_4-map is not necessarily a $T_{3\frac{1}{2}}$-map. For this, see the remark in this section.

We now give the definitions of submaps, compact maps [9] and locally compact maps [7].

Definition 2.5. The restriction of the projection $p : X \to B$ on a closed (resp. open, type G_δ, etc.) subset of the space X is called a *closed* (resp. *open, type G_δ*, etc.) *submap* of the map p.

Definition 2.6. (1) A projection $p : X \to B$ is called a *compact map* if it is perfect (i.e. it is closed and all its fibres $p^{-1}(b)$ are compact).
(2) A projection $p : X \to B$ is said to be a *locally compact map* if for each $x \in X_b$, where $b \in B$, there exists a nbd $W \in N(b)$ and a nbd $U \subset X_W$ of x such that $p' : X_W \cap \overline{U} \to W$ is a compact map, where p' is the restriction of p and $X_W \cap \overline{U}$ is the closure of U in X_W.

Note that a closed submap of a (resp. locally) compact map is (resp. locally) compact, and for a (resp. locally) compact map $p : X \to B$ and every $B' \subset B$ the restriction $p|X_{B'} : X_{B'} \to B'$ is (resp. locally) compact.

Definition 2.7. (1) For a map $p : X \to B$, a map $c(p) : c_pX \to B$ is called a *compactification* of p if $c(p)$ is compact, X is dense in c_pX and $c(p)|X = p$.
(2) A map $p : X \to B$ is called a *T_2-compactifiable map* (resp. *$T_{3\frac{1}{2}}$-compactifiable map*) if p has a compactification $c(p) : c_pX \to B$ and $c(p)$ is a T_2-map (resp. $T_{3\frac{1}{2}}$-map).

Remark: (1) The compactification of a map was studied by Pasynkov [7]. In James [5] Section 8, there are some basic study of compactifiable maps, but note that in [5] he uses a terminology "fibrewise compactification". For other study of compactifiable maps, see [1] and [6].
(2) Note that we must consider both T_2- and $T_{3\frac{1}{2}}$-compactifiable maps because, unlike the case of spaces, there exist T_2-compact maps which are not $T_{3\frac{1}{2}}$-maps ([4] 4.2 or [2] Example 3.4).

Definition 2.8. For the collection of fibrewise spaces $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$, the subspace $X = \{t = \{t_\alpha\} \in \prod\{X_\alpha : \alpha \in \Lambda : \ p_\alpha t_\alpha = p_\beta t_\beta \ \forall \alpha, \beta \in \Lambda\}$ of the Tychonoff product $\prod = \prod\{X_\alpha : \alpha \in \Lambda\}$ is called the *fan product* of the spaces X_α with respect to the maps p_α, $\alpha \in \Lambda$.

For the projection $pr_\alpha : \prod \to X_\alpha$ of the product $\prod$ onto the factor X_α, the restriction π_α on X will be called the projection of the fan product onto the factor $X_\alpha, \alpha \in \Lambda$. From the definition of fan product we have that, $p_\alpha \circ \pi_\alpha = p_\beta \circ \pi_\beta$ for every α and β in Λ. Thus one can define a map $p : X \to B$, called the *product* of the maps p_α, $\alpha \in \Lambda$, by $p = p_\alpha \circ \pi_\alpha$, $\alpha \in \Lambda$, and (X, p) is called the *fibrewise product space* of $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$.

Obviously, the projections p and $\pi_\alpha, \alpha \in \Lambda$, are continuous.

Proposition 2.9. Let $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$ be a collection of fibrewise spaces.
(1) If each p_α is T_i $(i = 0, 1, 2)$ (resp. functionally T_2), then the product p is also T_i $(i = 0, 1, 2)$ (resp. functionally T_2).
(2) If each p_α is a *surjective* T_3- (resp. $T_{3\frac{1}{2}}$-)map, then the product p is also a T_3- (resp. $T_{3\frac{1}{2}}$-)map
(3) If each p_α is a compact map, then the product p is a compact map.
(4) If each p_α is a T_2-compactifiable map, then the product p is a T_2-compactifiable map.

We shall conclude this section by defining the concept of b-filters (or tied filters) which plays an important role in this report.

Definition 2.10. ([5] Section 4.) For a fibrewise space X over B, by a *b-filter* (or *tied filter*) on X we mean a pair $(b, \mathcal{F})$, where $b \in B$ and $\mathcal{F}$ is a filter on X such that b is a limit point of the filter $p_*(\mathcal{F})$ on B. By an *adherence point* of a b-filter $\mathcal{F}$ $(b \in B)$ on X, we mean a point of the fibre X_b which is an adherence point of $\mathcal{F}$ as a filter on X.

3. SOME PROPERTIES

Definition 3.1. Let X be a topological space, and A a subset of X. We say that the *diameter of* A of the space X *is less than a family* $\mathcal{A} = \{A_s\}_{s \in S}$ of subsets of the space X, and we shall write $\delta(A) < \mathcal{A}$, provided that there exists an $s \in S$ such that $A \subset A_s$.

Definition 3.2. A T_2-compactifiable map p:$X \longrightarrow B$ is Čech-complete if for every $b \in B$, there exists a countable family $\{\mathcal{A}_n\}_{n \in N}$ of open (in X) covers of X_b with the property that every b-filter $\mathcal{F}$ which contains sets of diameter less than $\mathcal{A}_n$ for every $n \in \mathbf{N}$ has an adherence point.

Since the real line $\mathbf{R}$ with the usual topology is Čech-complete, $p : \mathbf{R} \to B$ is Čech-complete where B is a one-point space. All rational numbers $\mathbf{Q}$, as a subset of $\mathbf{R}$, is not Čech-complete, thus $p|\mathbf{Q}$ is not Čech-complete though $p|\mathbf{Q}$ is open and closed. But we have the following results.

Theorem 3.3. For a Čech-complete map $p : X \to B$, if F is closed subset of X, then $p|F : F \to B$ is Čech-complete.

Theorem 3.4. For a Čech-complete map $p : X \to B$, if G is a G_δ-subset of X and X is regular, then $p|G$: $G \to B$ is Čech-complete.

Theorem 3.5. Let $\{(X_n, p_n)|n \in N\}$ be a countable family of fibrewise spaces and $(\prod_B X_n, p)$ be the fibrewise product space. If every p_n is surjective Čech-complete , then p is Čech-complete.

Theorem 3.6. Let a fibrewise map $\lambda : (X,p) \to (Y,q)$ be a perfect map, and p and q be T_2-compactifiable maps. Then p is Čech-complete if and only if q is Čech-complete.

Theorem 3.7. Suppose that B is regular. For a T_2-compactifiable map $p : X \to B$, the following are equivalent:
(1) $p : X \to B$ is Čech-complete.
(2) For every T_2-compactipication $p' : X' \to B$ of p and each $b \in B$, X_b is a G_δ-subset of X'_b.
(3) There exists a T_2-compactipication $p' : X' \to B$ of p such that X_b is a G_δ-subset of X'_b for each $b \in B$.

Theorem 3.8. Every locally compact map, T_2-map is Čech-complete.

Definition 3.9. (James [5], Definitions 10.1 and 10.3) (1) Let (X,p) be a fibrewise spsce. The subset H of X is *quasi-open* (resp. *quasi-closed*) if the following condition is satisfied: for each $b \in B$ and $V \in N(b)$ there exists a nbd $W \in N(b)$ with $W \subset V$ such that whenever $p|K : K \to W$ is compact then $H \cap K$ is open (resp. closed) in K.
(2) Let a projection $p : X \to B$ be a T_2 map. The map p is a *k-map* if every quasi-closed subset of X is closed in X or, equivalently, if every quasi-open subset of X is open in X.

Theorem 3.10. Suppose that B is first countable and regular. Then a Čech-complete map $p : X \to B$ is a k-map.

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